CUSTOMER SUPPORT NOTE

The Calculation of Modulus of Subgrade Reaction for a "Winkler Spring" model

Note Number:	**CSN/LUSAS/1004**

This support note is issued as a guideline only.

LUSAS

Forge House, 66 High Street, Kingston upon Thames, Surrey, KT1 1HN, UK
Tel: +44 (0)20 8541 1999 Fax: +44 (0)20 8549 9399
Email: info@lusas.com www.lusas.com

Table of Contents

1. Introduction

In modelling soil, the approach most intuitive to structural engineers is where soil is represented by a subgrade reaction model ("Winkler springs").

However, using a spring stiffness to approximate soil behaviour is a major simplification. In general, Winkler models are reasonable if structural load effects (e.g. wall BM, SF, prop loads) are the main quantities of interest. Continuum models are required where soil movements (e.g. assessment of the likely damage to existing adjacent structures) are critical.

2. Description

2.1 Definition of modulus of subgrade reaction, k

The modulus of subgrade reaction is a conceptual relationship between applied pressure and deflection for a plate resting on an elastic support system. The defining equation is **$k=q/\delta$ where q=pressure, δ=deflection** and k is known as the modulus of subgrade reaction (units F/L^3) –sometimes called Winkler spring constant.

2.2 Problems obtaining k-values

In the field, the k-value is determined using data obtained from a plate loading test (typically 1ft by 1ft square). The load is applied to a stack of 1-inch thick plates until a specified pressure (q) or displacement (δ) is reached. Then $k=q/\delta$.

Unfortunately, for a specified displacement level, when plate size is increased, the computed k-value decreases. Teller and Sutherland (1943) investigated the effect of plate sizes on the k-value, concluding

1. It is necessary to limit the specified displacement to a magnitude comparable to those expected in the final foundation
2. It is of great importance to use a bearing plate of size appropriate to the foundations being assessed.

In essence, the engineer must know the size of the foundation (and expected deflection) in order to undertake an appropriate test to determine k-values to enter in his foundation calculations. This cyclical logic underlines the problems in determining k from field testing. Furthermore, it is difficult to make a plate-load test except for very small plates because of the reaction load required and therefore necessary to assess a suitable value from a small-scale test.

Thus plate loading tests (and theoretical derivations) show that as the loaded area increases, the computed k-value decreases. A k-value is not a fundamental property of soil; a k-value must be calculated specific to the foundation in question.

2.3 Notes on k-value for vertical springs representing subgrade

Terzaghi (1955) suggested relating the k-value derived from a plate loading test to the k-value for a "real" foundation in a number of formulae considering both cohesive and cohesionless soils. While this is useful, k-value derived from a plate test is found *also* to be sensitive to:

1. Load (or displacement) magnitude
2. Moisture content
3. Loading rate in cohesive saturated soils

Nonetheless Terzaghi's fomulae and typical values are still widely used and are outlined below:

$\bar{k}_{s1}$ = modulus of vertical subgrade reaction for a square plate 1ft x 1ft

This value is taken from field tests (can compare to typical values for reassurance), then converted to k_{s1} and finally k_s for use in design calculations.

2.3.1 Stiff clays

Typical values (after Terzaghi, 1955)

	Stiff	Very stiff	Hard
Undrained shear strength, C_u	100-200 kPa	200-400 kPa	>400 kPa
$\bar{k}_{s1}$ = range	15-30 MN/m³	30-60 MN/m³	>60 MN/m³
$\bar{k}_{s1}$ = recommended	23 MN/m³	45 MN/m³	90 MN/m³

$$k_s = k_{s1}\frac{1}{B(ft)} \equiv k_{s1}\frac{0.305}{B}$$

$$k_{s1} = \bar{k}_{s1}\left[\frac{L(ft)+0.5}{1.5L(ft)}\right] \equiv \bar{k}_{s1}\left[\frac{L+0.152}{1.5L}\right]$$

$k_h = k_s$

2.3.2 Cohesionless soils

Typical values (after Terzaghi, 1955)

	Loose	Medium	Dense
$\bar{k}_{s1}$ = dry/moist	5-20 MN/m³	20-90 MN/m³	90-300 MN/m³
$\bar{k}_{s1}$ = recommended	12 MN/m³	40 MN/m³	150 MN/m³
$\bar{k}_{s1}$ = submerged	8 MN/m³	25 MN/m³	100 MN/m³

$$k_s = k_{s1}\left[\frac{B(ft)+1}{2B(ft)}\right]^2 \equiv k_{s1}\left[\frac{B+0.305}{2B}\right]^2$$

$$k_{s1} = \bar{k}_{s1}$$

$k_h = m_h . z$

2.4 Relating k to E

Terzaghi's formula are still widely used but there has been interest in other possible means of determining k. Subsequent work, notably by Vesic (1963, 1970) has suggested relating k to E. Timoshenko & Goodier (1970) give a solution for the average displacement of a square flexible footing resting on a homogeneous isotropic linear elastic solid, carrying a vertical load. From it we deduce

$$k = \frac{q}{\delta} = \frac{E}{h}\frac{1}{0.95(1-\nu^2)} \text{ (for a loaded plate of side } h\text{)}$$

Again this indicates that k is not constant but depends on the size of the loaded area. Furthermore an appropriate k-value for a foundation analysis would depend on the loading location (edge of foundation, corner of foundation etc).

2.5 Other approximations

Since values of E are often unavailable, other approximations are also useful. Section 9-6 of JE Bowles' "Foundation Analysis and Design" 5th Edition (McGraw-Hill) entitled "modulus of subgrade reaction" includes approximations based on allowable bearing capacity, general solutions for horizontal/lateral modulus of subgrade reaction and a worked example. Based on allowable bearing pressure, qa:

ks=40*safety factor*qa

where safety factor would typically be 3 and qa from the ultimate settlement of about 0.0254m

e.g. ks=120*qa=40*q_ult

Note that site-specific data is always essential when considering soil parameters and a geotechnical expert should be consulted.

2.6 Fundamental problem with k

In the Winkler model, displacement at a point is proportional to the applied pressure at that point (δ =q/k), and this is visualised as springs of stiffness, k. There is no shear transmission between adjacent springs.

In reality the deflection of a point in the subgrade occurs not just as a result of the stress acting at that particular point but is influenced to a progressively decreasing extent by stresses at points distant. By not replicating this fundamental behaviour, the Winkler model is lacking and attempts to determine suitable values for k point to the deficiency in the concept.

The Timoshenko & Goodier (1970) solution for displacement of a square flexible footing carrying a vertical load P over an area A on a homogeneous isotropic linear elastic (HILE) solid also indicates that k is not constant but depends on the size of the loaded area. (see above)

We can conclude that a HILE solid needs at least 2 parameters to define it (E and v or G and K) so attempting to describe behaviour with just one parameter can be expected to lead to difficulties – in particular when attempting to select an appropriate value for a given practical problem. Being thus dependant on the size (and shape) of the structure being considered, subgrade modulus cannot be regarded as a fundamental material parameter.

2.7 Limitations & uses

The limitations of the Winkler (subgrade reaction) approach may be summarised:

- no prediction of soil movements at a distance from the foundation element is given.
- no shear transmission between adjacent springs, therefore no prediction of differential settlement (no "dished" profiles under uniform load)
- difficulty determining spring stiffness (k) leading to uncertainty in predicted total or average settlements

Terzaghi (1955) stated that 'the theories of subgrade reaction should not be used for the purpose of estimating settlement or displacements'. However:

- For an infinite beam under a line load settlements are proportional to $k^{-3/4}$, while bending moments are proportional to $k^{-1/4}$. Therefore BM not so sensitive to inaccuracy in k-value.
- Westergaard (1926) showed increase of k-value 4:1 = only minor changes in critical stresses for pavement design.
- Vesic (1961) suggested that it was possible to select a k-value so as to obtain a good approximation of both bending stresses and deflections of a beam resting on a soil, provided the beam is sufficiently long.

In general, continuum models are far better suited to the prediction of ground movements adjacent/ under the structure and remote from the structure. Winkler is still widely used in the determination of structural load effects.

2.8 Beam on elastic foundations

A general solution for a "Beam on elastic foundations" was derived using the Winkler spring analogy long before the advent of digital computers.

From beam theory

$M=EId^2v/dx^2$

$S=dM/dx$

From vertical equilibrium on a small length of beam

$dS/dx + Bq = Bkv$

From this is derived the governing equation

$$\mathbf{d^4v/dx^4 + Bk.v/EI = Bq/EI}$$

The general solution to this equation (Hetenyi, 1946) is

$V = C1\ e^{\lambda x} \cos\lambda x + C2\ e^{\lambda x} \sin\lambda x + C3\ e^{-\lambda x} \cos\lambda x + C4\ e^{-\lambda x} \sin\lambda x$

Where $\lambda = \sqrt[4]{\frac{Bk_s}{EI}}$, a relative stiffness term crucial to behaviour. C1, C2, C3, C4 are constants determined by the boundary conditions. $1/\lambda$ has units of length and defines a **"characteristic length"** for the problem.

For an infinite beam under a line load all quantities are negligible when $\lambda x > 1.5\pi$ hence

1. there is no interaction when loads are further apart than $x=1.5*\pi/\lambda$ and
2. Hetenyi infinite beam solution is valid when $L > 3*\pi$

2.9 Laterally loaded piles

The governing equation of a laterally loaded pile: $\mathbf{d^4v/dx^4 + Bk.v/EI = 0}$

An analytical solution similar to the classic BOEF solution by Hetenyi may be found. The main differences are that the pile is vertical, load may only be applied at the head, and the elastic (soil) medium exists on both sides. Assuming that the pile is long and the soil modulus is independent from depth, the pile deflection is of the form

$V = C1\ e^{-\beta x} \cos\beta x + C2\ e^{-\beta x} \sin\beta x$

β is a relative stiffness term crucial to behaviour. C1 and C2 are constants determined by the boundary conditions. $1/\beta$ has units of length and defines a **"characteristic length"** for the problem.

3. References

Notes from the University of California, San Diego:
http://geotechnic.ucsd.edu/se243/notes1.pdf

There is also an FD solution for a beam on an elastic foundation "online":
http://geotechnic.ucsd.edu/footing/footing.html

BOWLES, J.E. (1997). "Foundation analysis and design". 5th Edition, McGraw-Hill Inc.

SIMONS, N.E. and MENZIES, B.K. (1977). "A short course in foundation engineering". Butterworth & Co (Publishers) Ltd.

CERNICA, J.N. (1995). "Geotechnical Engineering: Foundation Design". 1st Edition, Wiley.

WINTERKORN, H.F. and FANG, H.Y. (1975). "Foundation Engineering Handbook". Van Nostrand Reinhold Company, pp 567.