CUSTOMER SUPPORT NOTE

Plastic properties for stress resultant beam model (model 29)

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This support note is issued as a guideline only.

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1. Introduction

The stress resultant beam model is based on a yield surface, which has been derived for **rectangular** solid sections and **circular hollow** sections. Other cross-sections may be treated as a combination of these. The yield criteria have been derived using the following initial assumptions:

- The von Mises yield criterion is used as the basis of the model.
- The stress-strain curve is linear elastic/perfectly plastic (i.e. zero hardening)
- Plastification is an abrupt process with the whole cross-section transformed from an elastic to fully plastic stress state.
- The fully plastic torsional capacity is constant.
- Transverse shear distortions are neglected.

Further information on the governing equations of this model can be found in Volume 1 of the LUSAS Theory Manual under section 4.3.2.

Model 29 requires the specification of the following plastic section properties:

- Plastic cross-section area (A_{p})
- Plastic modulii for bending about the y axis (Z_{yyp})
- Plastic modulii for bending about the z axis (Z_{yyp})
- Plastic modulii for torsion about the y axis (Z_{yp})
- Plastic modulii for torsion about the z axis (Z_{zp})

Note that plastic area for shear (S_{p}), is not actually used in the material formulation and therefore does not need to be specified (i.e. set it to 0). It is planned that this parameter will be removed from the user interface in future releases of LUSAS

Finally, the following material properties need to be specified under the plastic material interface:

- Uniaxial yield stress (s_o)
- The section shape on which the material model is formulated, either rectangular solid section or circular hollow section

This support note deals with the theory and calculation of the plastic section properties $\mathbf{Z_{yyp}}$, $\boldsymbol{Z_{zzp}}$, $\boldsymbol{Z_{yp}}$ and $\boldsymbol{Z_{zp}}$.

Please note that Model 29 assumes an abrupt change from the elastic to fully plastic stress state when the yield criteria is met.

2. Plastic modulii for moment [1]

First consider a beam with a rectangular solid section, of elasto-plastic material subjected to a bending moment M_{zz} about the local z axis.

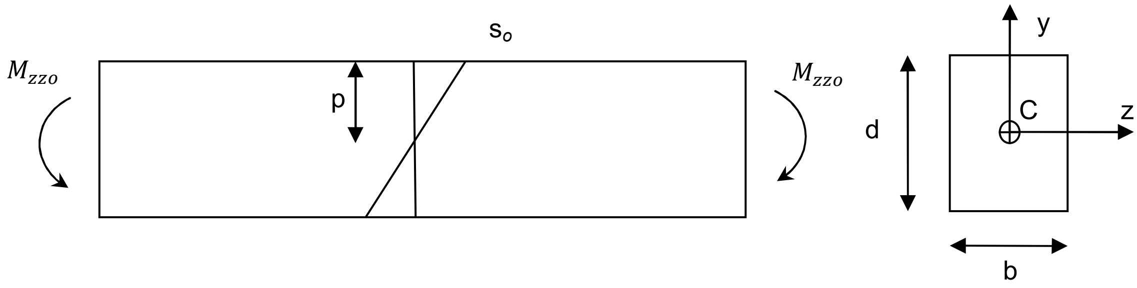

Figure 1

As M_{zz} increases, the stress on the most extreme fibre, at distance *p* from the neutral axis (NA) C, can increase to a maximum of s_o, the material yield stress. This moment is termed the yield moment and given about the z-axis as,

$$M_{zzo} = \frac{s_o \mathrm{I_{zz}}}{p} = s_o S_{zz}$$

Where $\mathrm{I_{zz}}$ and $S_{zz} = \frac{\mathrm{I_{zz}}}{p}$ are the second moment of area and the elastic section modulus respectively, both about the z axis. In addition, a similar expression could be given for the yield moment about y.

As the moment is increased past M_{zzo}, the strains in the beam will continue to increase past the yield strain, but due to the elastic-perfectly plastic condition, the maximum stress cannot exceed the yield stress. Instead, the yielded zone will increase until the stress throughout the entire section has reached s_o, as shown below.

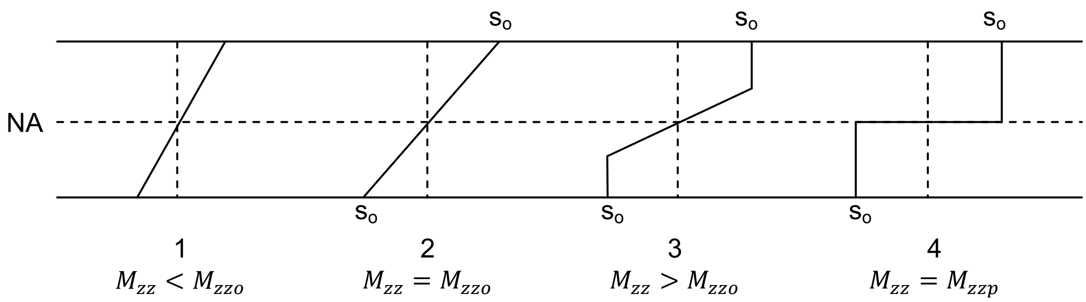

Figure 2

The bending moment corresponding to the final idealised stress distribution (in item 4 of Figure 2) is represented by the plastic moment M_{zzp}, which can be found by integrating over the cross-section area.

$$M_{zzp} = \int_A \sigma y dA$$

However at this instant, every point on the cross-section carries a tensile stress s_o above the NA and $-s_o$ below the NA, so the above expression can be reduced to

$$M_{zzp} = s_o(\bar{y}_1 A_1) - s_o(-\bar{y}_2 A_2)$$

Where A_1 and A_2 are the cross sectional areas above and below the neutral axis, and $\bar{y}_1$ and $\bar{y}_2$ are the distances from the NA to the centroids of A_1 and A_2.

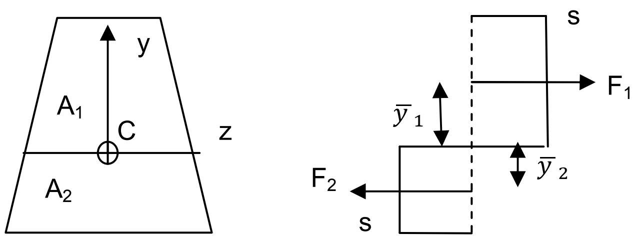

Figure 3

For pure bending, the resultant force acting on the section is zero (i.e. $F_1 = \mathrm{F}_2$), so under fully plastic conditions,

$$\mathrm{s_o} A_1 = s_o \mathrm{A}_2$$
$$A_1 = \mathrm{A}_2$$

Therefore, given that that the total area A, is equally divided between A_1 and A_2, then the fully plastic moment about the z axis is,

$$M_{zzp} = \frac{s_o A(\bar{y}_1 + \bar{y}_2)}{2} = s_o Z_{zzp}$$

Where,

$$Z_{zzp} = \frac{A(\bar{y}_1 + \bar{y}_2)}{2}$$

And similarly about the y-axis,

$$M_{yyp} = \frac{s_o A(\bar{z}_1 + \bar{z}_2)}{2} = s_o Z_{yyp}$$

$$Z_{yyp} = \frac{A(\bar{z}_1 + \bar{z}_2)}{2}$$

Therefore, for a doubly symmetrical cross section such as the rectangular section shown in Figure 1, the fully plastic bending moments are simply given by

$$M_{zzp} = s_o \frac{A}{2}\left(\frac{d}{4} + \frac{d}{4}\right) = s_o \frac{Ad}{4} = s_o Z_{zzp}$$

$$M_{yyp} = s_o \frac{A}{2}\left(\frac{\mathrm{b}}{4} + \frac{\mathrm{b}}{4}\right) = s_o \frac{Ab}{4} = s_o Z_{yyp}$$

Where, the plastic section modulii for a rectangular section about the z and y-axes are,

$$Z_{zzp} = \frac{bd^2}{4}$$

And,

$$Z_{yyp} = \frac{db^2}{4}$$

Values of plastic bending modulii can be found for a range of commercially available beam sections in texts such as Roark Formulas for Stress and Strain [1] or the AISC Manual for Steel Construction [2].

3. Plastic modulii for torsion

As discussed in the LUSAS Theory Manual, Model 29's formulation uses the plastic modulii for torsion, Z_{yp} and Z_{zp}, to calculate the fully plastic torsional moment T_p,

$$T_p = s_o(Z_{zp} + Z_{zp})$$

Although the torsional modulii can be specified in two separate components, it is most common to calculate a single term i.e. Z_{Tp}. Therefore, since the two components are eventually added together anyway, it is appropriate to divide this value equally between the two components.

$$Z_{yp} = Z_{zp} = \frac{Z_{Tp}}{2}$$

The response of an elastic-perfectly plastic beam, subject to pure torsion can be illustrated in a similar way to the pure bending case above. The material will remain in the elastic range as long as the applied torque T is below the yield torque T_o, but first yield will occur at $T = T_o$ when the maximum shear stress $\hat{\tau}$ on the cross-section reaches 0.577s_o (i.e. from the Von Mises condition $\tau_y = \frac{s_o}{\sqrt{3}}$).

A theoretical way to compute the plastic modulii for torsion is explained below using the "Sand Heap Analogy".

Sand Heap Analogy [4]

The sand heap analogy is similar in concept to the more commonly known 'membrane' (or 'soap bubble') analogy for elastic torsion. As discussed in structural mechanics text books, the maximum shear stress of a cross-section can be calculated via the Prandtl stress function or described using the membrane analogy, which says that the shear stresses in a pure torsion problem are analogous to the slope of a membrane of the same shape as the beam cross-section, that is pinned at its edges and subjected to a uniform pressure underneath.

In addition, the theory says that the elastic torque, which is given by the following equation, is analogous to twice the volume under the membrane, since the stress function Φ corresponds to the membrane deflection, z at a point on the cross-section.

$$T = \iint \phi dxdy = 2\iint zdxdy$$

Because the stress function Φ is not usually known (except for example in the case of an elliptical section), the soap bubble analogy is primarily used to justify assumptions for an approximate calculation, for example with 'bars of narrow cross section' and 'thin walled tubes' (see text books [1,5] for details)

In a similar way, the sand heap analogy compares the stress function to a pile of dry sand, which of course is limited by instability at a given slope. This slope is analogous to the shear stress at yield, which for the fully plastic condition when $T = T_p$, is constant over the entire section. As such, because the torque is equal to twice the volume under the stress function, the fully plastic torque is also equal to twice the volume of the sand heap, which for a rectangular section is a pyramid.

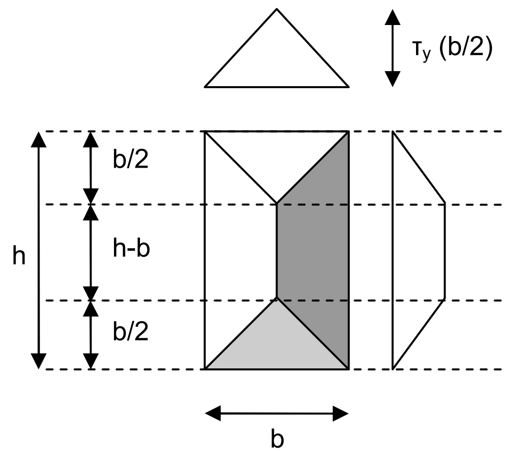

Figure 4

$$T_p = 2(\text{Volume under stress function } \emptyset)$$
$$= 2\left[\frac{b}{2}\left(\frac{b\ \tau_y}{2}\right)(h-b)\right] + \left[\frac{1}{3}\left(\frac{b\tau_y}{2}\right)b^2\right] = 2\ \tau_y \frac{b^3}{6}\left[\left(\frac{3}{2}\right)\left(\frac{h}{b}-1\right)+1\right]$$
$$= \tau_y \frac{b^3}{3}\left[\left(\frac{3}{2}\right)\left(\frac{h}{b}-1\right)+1\right]$$
$$= \tau_y Z_{Tp}$$

Where the fully plastic torsional modulii

$$Z_{Tp} = \frac{b^3}{3}\left[\left(\frac{3}{2}\right)\left(\frac{h}{b}-1\right)+1\right]$$

Therefore, unlike the elastic ‘soap bubble analogy’, where the ‘deflected surface’ (i.e. stress function) can be complex, the constant gradient (i.e. shear) condition in the sand heap analogy means that the volume and fully plastic torsional modulii can be calculated in a rather simpler approach.

A list of plastic torque values for a number of standard structural shapes is provided in Table 3.1 of ‘Ductile Design of Steel Structures’ [4], from which Z_T can be calculated by dividing by τ_y.

4. References

1. Mechanics of Materials, JM Gere, SP Timoshenko, Stanley Thornes, Fouth Edition, 1999
2. AISC Manual of Steel Construction, LRFD Second Edition, Chicago:American Institute for Steel Construction, 1992
3. Roark's Formulas for Stress and Strain, WC Young, RG Budynas, McGraw-Hill, Seventh Edition, 2002
4. Ductile Design of Steel Structures, M Bruneau, CM Uang, AS Whittaker, McGraw-Hill, 1998. *http://books.google.co.uk/books?id=cKs1W7be_w8C&pg=PA96&lpg=PA96&dq=%22plastic+torsion%22+handbook&source=web&ots=MVZBlSMn7_&sig=2A6zoKF1KMFfNGhamF59CqwBcZk&hl=en&sa=X&oi=book_result&resnum=1&ct=result#PPA100,M1*
5. Engineering Mechanics of Solids, EP Popov, Prentice Hall, Second Edition, 1999